

HYDRODYNAMIC AND ELASTOHYDRODYNAMIC LUBRICATION

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Summary

Moving surfaces in various machines and live organisms are exposed to external loading and are subjected to normal and frictional stresses which, to a certain extent, limit their useful life. In most cases these joints are lubricated one way or another. Therefore, the parameters such as lubricant viscosity, normal stresses, lubrication film thickness and contact friction are the keys to the longevity of the joints. That necessitates studying of lubricated contacts. Some history of the developments in the hydrodynamic and elastohydrodynamic lubrication (EHL) theories is presented. This essay covers the most fundamental problems of the EHL theory. Specifically, some new insights in the stress-induced lubricant degradation are presented, a simple steady isothermal problem for lightly loaded EHL contact with Newtonian lubricant is considered, some steady isothermal and thermal problems for line and point heavily loaded EHL contacts lubricated with Newtonian and non-Newtonian lubricants are studied, modeling of grease lubrication is considered. The approach employed to studying these problems is the analysis of the problem by analytical asymptotic techniques which is followed by numerical solution of the asymptotically simplified problem. The asymptotic approach to EHL problems allowed for development of an effective regularization numerical method for solution of the isothermal EHL problems in the original (non-asymptotic) formulations.

1. Introduction

Moving parts in machinery and live bodies involve different kind of lubricated joints. Lubrication of these joints serves several purposes: separation of contact surfaces, reduction of friction between the contact solids, reduction of wear, retardation of other forms of contact fatigue such as pitting, flaking, delamination, scuffing, etc. Therefore, it is crucial for machine design to be able to determine lubrication film thickness and contact friction. Lubricated joints involve contacts which can be classified as conformal and non-conformal contacts. Conformal contacts of solids are those in which the area of the contact is comparable in size with the sizes of the solids. Examples of such joints are represented by human and animal joints, journal and spherical bearings. On the other hand, joints involving non-conformal contacts are represented by roller and ball bearings, gears, etc. The size of the contact area in non-conformal contacts is much smaller than the curvature radii of the contact solids. Utilization of specific bearings or gears depends on the loading, lubrication, and environmental conditions. It means that some joints are lubricated by fluid lubricants while others by greases or solid lubricants. Usage of solid lubricants which are introduced into joints as thin coatings is typical for vacuum applications. The most often used solid lubricants are based on MoS₂ (molybdenum disulfide) and DLC (diamond-like-carbon) coatings. Greases are usually applied in cases when it is impractical to provide fluid lubrication due to the joint design and its application or economically ineffectiveness. Greases are made on the basis of different thickeners/soaps (aluminum, sodium, calcium, lithium, etc.) saturated with various oils. The most often choice for lubrication is fluid lubrication.

Lubricant behavior is controlled by its rheology which is the reflection of the lubricant structure. In most cases fluid lubricants are almost incompressible. The rheology of a lubricant is determined by the stress-strain relationship. The simplest lubricant rheology is Newtonian which is characterized by a linear dependence between shear stresses and strains. However, in most practical situations Newtonian rheology is just a simple

idealization. In practice, lubricants exhibit various features of non-Newtonian rheology. All greases have features of non-Newtonian rheology. Most of fluid lubricants also exhibit non-Newtonian behavior. Some rheological models of such lubricant behavior are represented by Ostwald-de Waele, Reiner-Philippoff-Carreau models and can be found in [Bair, 2007]. Contemporary formulated lubricants are base stock solutions of various additives. These additives serve such roles as viscosity improvers, corrosion inhibitors, EP (extreme pressure) additives used for wear protection under mixed and boundary lubrication conditions, etc. Lubricant viscosity is a part of any rheological model. In many lubricated joints pressure reaches high levels. Therefore, it is essential to know lubricant behavior at high stresses and under fast changing conditions. A significant progress in studying high pressure lubricant rheology and lubricant viscosity is done in [Bair, 2007].

For many lubricants their viscosity significantly increases with pressure and decreases with temperature. Contact solid surfaces moving with different linear speeds as well as high pressure are conducive for occurrence of high shear stresses. Shear stresses cause stress-induced lubricant degradation, raise of lubricant temperature, variation in lubricant friction which depends on the surface roughness and lubrication film thickness separating the contact surfaces, oxidation processes, etc. Lubricant degradation causes permanent loss of viscosity while lubricant oxidation usually increases its viscosity. This lubricant behavior is one of the manifestations of lubricant non-Newtonian rheology.

Depending on the loading and speed conditions, elastic moduli of the materials of contact solids as well as the contact size and lubricant viscosity a lubrication regime is lightly or heavily loaded. In lightly loaded lubricated contacts surface deformations are negligibly small and the situation is closely described by the behavior of lubricant between two moving rigid solids [Kudish and Covitch, 2010]. Such lubrication regimes are usually associated with hydrodynamic lubrication. In cases of heavily loaded lubricated contacts the situation is opposite. In the central part of the contact the pressure distribution is close to the one in a dry contact of the same solids (Hertzian pressure) and elasticity effects dominate the behavior of the lubricated contact while the film thickness between the contact surfaces is small and practically constant. These are usually referred to elastohydrodynamic lubrication (EHL) regimes. At the same time, in the zones close to contact boundaries the contribution of the lubricant flow and contact solid elasticity are comparable [Kudish and Covitch, 2010]. The conditions when the lubrication film thickness is comparable to the height of surface asperities are usually referred to as regimes of mixed friction. The lubrication conditions in which the most of the applied load is carried by contacts of surface asperities are called regimes of boundary friction.

The realization of the above lubrication regimes to a great extent depends on the amount of lubricant supplied to contact. If the amount of lubricant entering contact is relatively low then starved lubrication conditions occur. When the amount of lubricant supplied to contact is abundant a lubricated contact is in a fully flooded lubrication regime. The other contact parameter which affects lubrications regime is the lubricant temperature. The lubricant temperature is determined by the applied load, surface speeds, lubricant viscosity as a function of lubricant pressure and temperature, elastic and thermal solid

material parameters, and lubricant thermal parameters. The lubricant temperature, in turn, significantly affects friction in the contact. For relatively low temperatures a contact may be in a fully flooded lubrication regime while for high temperatures the same contact with the same lubricant supply may be in missed lubrication regime. Finally, contact friction is one of the main parameters affecting wear, pitting and other contact fatigue phenomena.

In certain cases lubricated contacts are involved in non-steady conditions and their lubricated contact parameters are generally involved in oscillations. The difference between a non-steady and the corresponding steady parameters for the same lubricated contact can be significant. For example, the ratio of the maxima of contact pressure in non-steady and steady contacts can exceed threefold. A great role in this difference between steady and non-steady lubricated contact parameters plays the inertia of the contact, i.e. the masses of the solids involved in the motion.

Often, real lubricated joints are idealized. For example, a lubricated contact of a roller with a bearing ring can be described as a contact of an infinite cylinder/roller made of an elastic material with an elastic half-plane, i.e. the EHL problem for such a situation can be considered in just two dimensions of the cross section of the roller perpendicular to its axis. Such a contact is called a line contact. Moreover, for non-conformal contacts the circular shape of the roller can be approximately replaced by the corresponding parabola. In case of a contact of a ball with a bearing ring the EHL problem can be reduced to a contact of an elliptical paraboloid made of an elastic material with an elastic half-space. Such a problem is usually called a point EHL contact. The case of a contact of gear teeth also can be reduced to a point EHL problem.

The foundations of the theory of hydrodynamic and elastohydrodynamic lubrication were made in ground breaking studies [Petrov, 1883], [Reynolds, 1884], [Sommerfeld, 1904], [Petrusevich, 1951], and [Ertel, 1945]. In particular, Reynolds presented a simple reduction of Navier-Stokes equations for a slow thin flow of viscous Newtonian fluids while Sommerfeld received a simple solution of a lubrication problem for a journal bearing. The next step in improving the lubrication theory for line contacts was done by Petrusevich who introduced elastic displacements of contact surfaces in the problem equations and for the first time solved the problem numerically. Ertel, in turn, offered a transparent approximate analytical procedure for calculation of lubrication film thickness in heavily loaded line contacts when lubricant viscosity (for Newtonian rheology) depends strongly on pressure. After that hydrodynamic and elastohydrodynamic lubrication problems enjoyed and continue enjoying close attention of many researchers. A further advancement in the EHL theory was made in [Jiang, Xiaofei et al., 1998] by introducing thermal effects and surface roughness in lubricated contacts. In a number of papers [Greenwood, 1972; Archard and Baglin, 1986] some advancements in Ertel-type approximate analytical solutions for line EHL problems were made. However, various numerical methods remain the prevalent approach to solution of EHL problems. Among these numerical methods there are studies based on the Newton-Raphson method [Hamrock, 1991], other numerical methods using the singularity in the integral equation for the gap [Evans and Hughes, 2000], multilevel multi-grid methods [Venner, and Lubrecht, 2000], fast Fourier based methods [Jiang, Xiaofei et al., 1998]. Furthermore, there was developed a method [Kudish and Covitch,

2010] which synthesizes the benefits of analytical and numerical approaches to solution of EHL problems. The approach is based on combination of regular and matched asymptotic expansions and specialized numerical methods.

2. Stress-Induced Lubricant Degradation

Considering stress-induced degradation of polymer additives/viscosity modifiers (VM) to lubricants it is important to realize that the chemical and geometrical composition of the polymer additive molecules can be different. For example, additive molecules can be represented by polymer molecules with linear structure, star polymer molecules, etc. Here only the case of linear polymer molecules is considered. These polymer molecules as well as star polymer molecules are studied in detail in [Kudish and Covitch, 2010].

Let a lubricant formulated with VM represented by linear polymer molecules is involved in a cyclic motion process during which polymer molecules experience shear stresses applied to them. For simplicity we will assume that the polymer molecules are stretched along the lubricant flow streamlines described by the equation

$$\frac{d\vec{x}}{dt} = \vec{u}(t, \vec{x}), \quad (1)$$

where \vec{u} is the lubricant fluid velocity at the point \vec{x} at the time t . Let us derive the kinetic equation describing the process of stress-induced degradation. It is necessary to introduce the density of the statistical distribution $W(t, \vec{x}, l)$ of polymer molecular weight in such a way that $W(t, \vec{x}, l)\Delta l\Delta v$ is the weight of polymer molecules with the chain lengths between l and $l + dl$ located in the volume Δv centered at the point \vec{x} at the time moment t . Assuming that the monomer molecular weight is w_m it is obvious that $W(t, \vec{x}, l) = lw_m n(t, \vec{x}, l)$, where $n(t, \vec{x}, l)$ is the density of the distribution of the number of polymer molecules with the chain lengths between l and $l + dl$ located in the volume Δv centered at the point \vec{x} at the time moment t .

Polymer degradation is caused by polymer molecule scission. Let us assume that at any time moment a polymer molecule can undergo scission at just one point, i.e. it can be broken in just two parts. Let $0 \leq R(t, \vec{x}, l) \leq 1$ be the probability of a polymer molecule of length l located at the point \vec{x} at the time t to be broken into two parts while $p_c(t, \vec{x}, l, L)$ be the density of the conditional probability for a polymer molecule of chain length L to be broken into two parts of lengths l and $L - l$. Obviously,

$$p_c(t, \vec{x}, l, L) \geq 0, \quad \int_0^L p_c(t, \vec{x}, l, L) dl = 1 \quad \text{for } L > 0. \quad \text{Suppose that the characteristic time of}$$

polymer scission is τ_f . Taking into account the motion of the fluid volume along a flow streamline from point \vec{x} to point $\vec{x} + \vec{u}\Delta t$ over the time period Δt and the balance of the polymer molecule weight in the process of this motion and stress-induced scission allowing $\Delta t \rightarrow 0$ one obtains the initial-value problem for the kinetic equation of polymer scission [Kudish and Covitch, 2010]

$$\rho \left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial \bar{x}} \right) \frac{W}{\rho} = \frac{2l}{\tau_f} \left\{ \int_l^\infty R(t, \bar{x}, L) p_c(t, \bar{x}, l, L) W(t, \bar{x}, L) \frac{dL}{L} - R(t, \bar{x}, l) W(t, \bar{x}, l), \right\}, \quad (2)$$

$$W(0, \bar{x}(0), l) = W^0(l),$$

where differentiation is done along the flow stream lines (see (1)), ρ is the fluid density, τ_f is the characteristic time of one act of polymer molecule fragmentation, and $W^0(l)$ is the known initial density of molecular weight distribution at the initial point $\bar{x}(0)$ on the flow streamline at time $t = 0$.

The probability of scission $R(t, \bar{x}, l)$ depends on the nature of polymer additive material (i.e. the polymer molecule bond dissociation energy U , bead radius a_* , and bond length l_*) as well as the lubricant temperature T and applied stress μS (S is the shear rate). Making an assumption that a polymer molecule breaks only if the deformation energy supplied by the lubricant shear stresses applied to the side surface of a polymer molecule and accumulated by this molecule per one molecule bond is above the dissociation energy for this molecule bond U/N_A (N_A is the Avogadro number, $N_A = 6.022 \times 10^{23}$ mole⁻¹) leads to a relatively simple derivation of the formula [Kudish and Covitch, 2010]

$$R(t, \bar{x}, l) = 1 - \left(\frac{l}{L_*} \right)^{\frac{2\alpha U_A}{kT}} \exp \left[-\frac{\alpha U_A}{kT} \left(\frac{l^2}{L_*^2} - 1 \right) \right] \text{ if } l > L_*; \quad R(t, \bar{x}, l) = 0 \text{ if } l \leq L_*, \quad (3)$$

$$L_* = \sqrt{\frac{\mu_a}{\mu}} L_0, \quad L_0 = \sqrt{\frac{U_A}{C \alpha l_*^2 \mu_a S}}, \quad U_A = \frac{U}{N_A}, \quad (4)$$

Where k is Boltzmann's constant, $k = 1.38 \times 10^{-23}$ J / K, μ_a is the lubricant viscosity before scission, C and α are dimensionless constants which should be determined by comparison of theoretical and experimental data.

Derivation of the density of the conditional probability of scission $p_c(t, \bar{x}, l, L)$ is also energy based and can be obtained in the form [Kudish and Covitch, 2010]

$$p_c(t, \bar{x}, l, L) = \ln 2 \frac{4|L-2l|}{L^2} \exp \left[-\ln 2 \frac{4l(L-l)}{L^2} \right]. \quad (5)$$

It is remarkable that the density of the conditional probability of scission $p_c(t, \bar{x}, l, L)$ is independent of \bar{x} and t as well as any adjustable constants.

According to empirical Huggins and Mark-Houwink formulas [Billmeyer, 1962; Crespi, Vassori, Slisi, 1977] the lubricant viscosity μ depends on the additive molecular weight distribution as follows

$$\mu = \mu_a \frac{1 + c_p[\eta] + k_H(c_p[\eta])^2}{1 + c_p[\eta]_a + k_H(c_p[\eta]_a)^2}, \quad [\eta] = k' M_W^\beta, \quad M_W^\beta = \int_0^\infty w^\beta W(t, \bar{x}, w) dw / \int_0^\infty W(t, \bar{x}, w) dw, \quad (6)$$

where c_p and $[\eta]$ are the polymer concentration and the intrinsic viscosity, respectively, $[\eta]_a$ is the intrinsic viscosity before scission, k_H is the Huggins empirical constant, k' and β are the Mark-Houwink empirical constants, $w = w_m l$ is the polymer chain molecular weight.

It can be shown that the solution of the problem (1)-(6) exists and is unique and possesses certain properties [Kudish and Covitch, 2010]. For example, in the process of stress-induced degradation of a lubricant in an isolated system the number of polymer molecules increases while the average polymer chain length decreases. Equation (2) can be numerically integrated with respect to time t along the flow streamlines using the rectangle rule for calculation of integrals. After each step in time the lubricant viscosity μ should be recalculated according to Eqs. (6) which trigger updating the probability of scission $R(t, \bar{x}, l)$ from formulas (3) and (4). Details of the numerical solution can be found in [Kudish and Covitch, 2010]. An example of such numerical solution and its comparison with the independently obtained experimental data is presented in Figures 1 and 2. The numerical and experimental data with the same initial molecular weight distribution $W^0(l)$ were obtained for a lubricant with OCP10 viscosity improver. The following parameters were accepted for numerical calculations

$$U = 347 \text{ kJ/mole}, \quad l_* = 0.154 \text{ nm}, \quad a_* = 0.374 \text{ nm}, \quad T = 310\text{K}, \\ \mu_a = 0.00919 \text{ Pa}\cdot\text{s}, \quad S = 5000 \text{ s}^{-1} \text{ and } C = 0.044, \quad \alpha = 0.008.$$

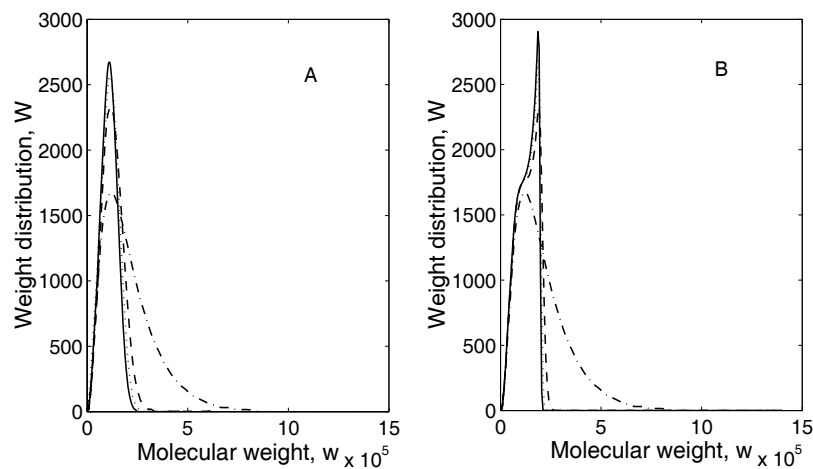


Figure 1. Polymer molecular weight distributions at different time moments during testing of the lubricant with OCP10 VM (A) and obtained from numerical modeling (B)

for $U = 347$ kJ/ mole, $C = 0.044$, and $\alpha = 0.008$: dashed-dotted curve – initial molecular distribution, dashed curve – after 30 cycles, dotted curve – after 100 cycles, solid curve – after 250 cycles (after Kudish, Airapetyan, and Covitch [Kudish, Airapetyan, and Covitch, 2003]). Reprinted with the permission from STLE.

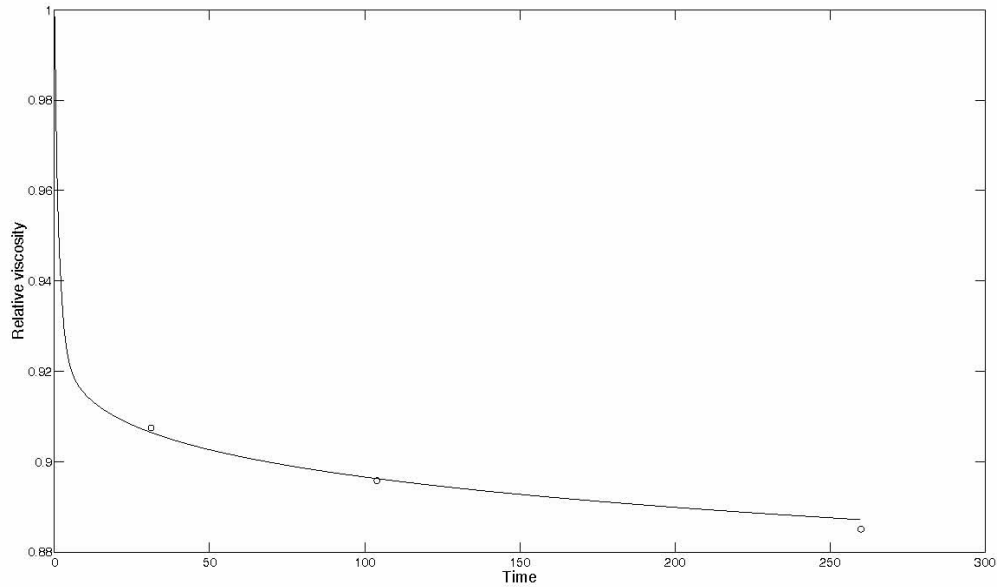


Figure 2. Loss of the lubricant viscosity μ caused by polymer molecule degradation versus number of cycles for the lubricant with OCP10 VM obtained for $U = 347$ kJ/mole, $C = 0.044$, and $\alpha = 0.008$. Circles indicate the experimentally measured relative viscosity of the lubricant with OCP10 VM.

Figure 1 shows a good agreement between the numerical and experimental molecular weight distributions while Figure 2 shows the permanent loss of about 10% of lubricant viscosity as a result of stress-induced lubricant degradation. Modeling of stress-induced lubricant degradation with star polymer additive is more complex and can be reduced to solution of a system of a number of somewhat similar to (2) integro-differential equations [Kudish and Covitch, 2010]. The number of the equations in the system is equal to the maximum number of arms of star polymer molecules in the lubricant solution. The comparison of the numerical solutions of the system provides a good agreement with the test data. An isothermal problem for a heavily loaded line EHL contact with degrading lubricant is considered in [Kudish and Covitch, 2010]. The structure of the considered additive was linear.

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Biographical Sketch

Ilya I. Kudish was born in Russia. In 1973 he graduated Moscow Institute of Physics and Technology and in 1981 he received his Ph.D. degree in Mathematics and Physics from St. Petersburg Polytechnic University, St. Petersburg, Russia. In 1991 I.I. Kudish and his family immigrated to the USA. Since 1994 he works at Kettering University first as an Associate and 2000 as Full Professor of Mathematics. In 2004 Ilya I. Kudish was elected Fellow of the ASME, in 2005-2011 served as an Associate Editor of the ASME Journal of Tribology. Also, over the years he served as a reviewer of several research journals in the field of tribology. Ilya I. Kudish published over 100 research papers in the fields of contact problems of elasticity, fracture mechanics, elastohydrodynamic lubrication, lubricant degradation, and fatigue modeling. He developed distinct asymptotic approaches to various problems tribology as well as the theory of lubricant degradation and some models of contact and structural fatigue. In 2010 he together with M.J. Covitch published a monograph “Modeling and Analytical Methods in Tribology”, CRC Press.